Viscosity solutions of first order Hamilton-Jacobi equations in locally compact Hadamard spaces

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We propose a novel approach to study first order Hamilton-Jacobi equations in a certain class of metric spaces called locally compact Hadamard spaces.

A metric space (X, d) is said to be a *Hadamard space* if, roughly speaking, it is a complete geodesic space and its geodesic triangles are "thinner" than the triangles of the Euclidean plane $\mathbb{R}^2$. They can be seen as a generalization of Hilbert spaces or Hadamard manifolds. Typical examples of Hadamard spaces include Hilbert spaces, metric trees and networks obtained by gluing a finite number of half-spaces along their common boundary.

Although Hadamard spaces are not manifolds in general, they carry a solid first order calculus resembling that of a Hilbert space. For example, a notion of tangent cone is well defined at each point of X. The tangent cone is the metric counterpart of the tangent space in Riemannian geometry or the Bouligand tangent cone in convex analysis. Furthermore, a notion of differential is well defined for any real-valued function $u : X \to \mathbb{R}$ that is Lipschitz and can be represented as a difference of two semiconvex functions (Lipschitz and DC functions in short). We propose to exploit all this additional structure that Hadamard spaces enjoy to study stationary and time-dependent first order Hamilton-Jacobi equation in them. In particular, we want to recover the main features of viscosity theory: *the comparison principle and Perron's method.*

In this talk, we give the main hypotheses we require on the Hamiltonian in this setting. Furthermore, we define the notion of viscosity using test functions that are Lipschitz and DC. Moreover, we show that we obtain the comparison principle using the variable doubling technique. Finally, we derive existence of the solution from the comparison principle using Perron's method in a similar manner as in the classical case of $X = \mathbb{R}^N$.

The main results are the following. First, we show that this new notion of viscosity coincides with the classical one in $\mathbb{R}^N$ by studying the examples of Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Isaacs' equations. Furthermore, we prove existence and uniqueness of the solution of Eikonal-type equations posed in networks that can result from gluing half-spaces of different Hausdorff dimension.